A spinor-like representation of the contact superconformal algebra $K'(4)$

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In this work we construct an embedding of a nontrivial central extension of the contact superconformal algebra $K'(4)$ into the Lie superalgebra of pseudodifferential symbols on the supercircle $S^{1|2}$. Associated with this embedding is a one-parameter family of spinor-like tiny irreducible representations of $K'(4)$ realized just on 4 fields instead of the usual 16.

I. Introduction

Recall that a *superconformal algebra* is a simple complex Lie superalgebra, such that it contains the centerless Virasoro algebra (i.e. the Witt algebra) $Witt = \oplus_{n\in\mathbb{Z}}\mathbb{C}L_n$ as a subalgebra, and has growth 1. The $\mathbb{Z}$-graded superconformal algebras are ones for which adL_0 is diagonalizable with finite-dimensional eigenspaces; see Ref. 1. In general, a superconformal algebra is a subalgebra of the Lie superalgebra of all derivations of $\mathbb{C}[t, t^{-1}] \otimes \Lambda(N)$, where $\Lambda(N)$ is the Grassmann algebra in N odd variables.

The Lie superalgebra $K(N)$ of contact vector fields with Laurent polynomials as coefficients is characterized by its action on a contact 1-form (Refs. 1, 2, 3, and 25); it is isomorphic to the $SO(N)$ *superconformal algebra* (Ref. 4). $K(N)$ is simple except when $N = 4$. In this case $K'(4) = [K(4), K(4)]$ is simple. Note that $K'(N)$ is spanned by 2^N fields. It was discovered independently in Ref. 3 and Ref. 5 that the Lie superalgebra of contact vector fields with polynomial coefficients in 1 even and 6 odd variables contains an exceptional simple Lie superalgebra (see also Ref. 2, Refs. 6, 7, and Refs. 23, 24). In Ref. 3 the exceptional superconformal algebra CK_6 was discovered as a subalgebra of $K(6)$, and it was shown that the derived Lie superalgebra of divergence-free derivations of $\mathbb{C}[t, t^{-1}] \otimes \Lambda(2)$, which is spanned by 8 fields, can be realized inside $K(4)$ using the construction of CK_6 inside $K(6)$.

Note that a Lie algebra of contact vector fields can be realized as a subalgebra of Poisson algebra; see Ref. 8. The Poisson algebra of formal Laurent series on $\dot{T}^*S^1 = T^*S^1 \setminus S^1$ has

a well-known deformation, that is the Lie algebra R of pseudodifferential symbols on the circle. The Poisson algebra can be considered to be the semiclassical limit of R; see Refs. 9, 10, 11, and 12.

In this work we define a family $R_h(N)$ of Lie superalgebras of pseudodifferential symbols on the supercircle $S^{1|N}$, where $h \in]0,1]$, which contracts to the Poisson superalgebra.

For each h we construct an embedding of a central extension $\hat{K}'(4)$ into $R_h(2)$. These central extensions are isomorphic to one of 3 independent central extensions, which are known for $K'(4)$ (Refs. 1, 2, 13 and 14). The corresponding central element is $h \in R_h(2)$. The elements of embeddings of $\hat{K}'(4)$ are power series in h; considering their limits as $h \to 0$, we obtain an embedding of $K'(4)$ into the Poisson superalgebra.

The idea of our construction is as follows. We consider the Schwimmer-Seiberg's deformation $S(2,\alpha)$ of the Lie superalgebra of divergence-free derivations of $\mathbb{C}[t,t^{-1}] \otimes \Lambda(2)$ (Refs. 15 and 1) and observe that the exterior derivations of $S'(2,\alpha)$ form an $\mathfrak{sl}(2)$ if $\alpha \in \mathbb{Z}$. The exterior derivations of $S'(2,\alpha)$ for all $\alpha \in \mathbb{Z}$ generate a subalgebra of the Poisson superalgebra isomorphic to the loop algebra $\tilde{\mathfrak{sl}}(2)$ [$\mathfrak{sl}(2)$ corresponds to $\alpha = 1$]. We prove that the family $S'(2,\alpha)$ for all $\alpha \in \mathbb{Z}$ and $\tilde{\mathfrak{sl}}(2)$ generate a Lie superalgebra isomorphic to $K'(4)$. The similar construction for each $h \in]0,1]$ gives an embedding of a nontrivial central extension of $K'(4)$:

$$\hat{K}'(4) \subset R_h(2). \tag{1.1}$$

It is known that the Lie algebra R has two independent central extensions; see Refs. 9, 10, and 11. Accordingly, there exist, up to equivalence, two nontrivial 2-cocycles on its superanalog $R_{h=1}(N)$. The 2-cocycle on $K'(4)$, which corresponds to the central extension $\hat{K}'(4)$ is equivalent to the restriction of one of the 2-cocycles on $R_{h=1}(2)$.

Finally, the embedding (1.1) for $h = 1$ allows us to define a new one-parameter family of tiny irreducible representations of $\hat{K}'(4)$. Recall that there exists a two-parameter family of representations of $K'(N)$ in the superspace spanned by 2^N fields. These representations are defined by the natural action of $K'(N)$ in the spaces of "densities"; see Ref. 1.

We obtain representations of $\hat{K}'(4)$, where the value of the central charge is equal to 1, realized on just 4 fields, instead of the usual 16.

II. Superconformal algebras

In this section we review the notion of a superconformal algebra and give the necessary

definitions.

A *superconformal algebra* is a complex Lie superalgebra $\mathfrak{g}$ such that

1) $\mathfrak{g}$ is simple,

2) $\mathfrak{g}$ contains the Witt algebra $Witt = der\mathbb{C}[t, t^{-1}] = \oplus_{n\in\mathbb{Z}}\mathbb{C}L_n$ with the well-known commutation relations

$$[L_n, L_m] = (n-m)L_{n+m} \tag{2.1}$$

as a subalgebra,

3) adL_0 is diagonalizable with finite-dimensional eigenspaces:

$$\mathfrak{g} = \oplus_j \mathfrak{g}_j, \mathfrak{g}_j = \{x \in \mathfrak{g} \mid [L_0, x] = jx\}, \tag{2.2}$$

so that $dim\mathfrak{g}_j < C$, where C is a constant independent of j; see Ref. 1. The main series of superconformal algebras are $W(N)$ $(N \geq 0)$, $S'(N, \alpha)$ $(N \geq 2)$, and $K'(N)$ $(N \geq 1)$; see Refs. 1 and 25. The corresponding central extensions were classified in Ref. 1; see also Refs. 2, 13, 14 and 16.

The superalgebras $W(N)$. Consider the superalgebra $\mathbb{C}[t, t^{-1}] \otimes \Lambda(N)$, where $\Lambda(N)$ is the Grassmann algebra in N variables $\theta_1, \ldots, \theta_N$. Let p be the parity of the homogeneous element. Let $p(t) = \bar{0}$ and $p(\theta_i) = \bar{1}$ for $i = 1, \ldots, N$. By definition $W(N)$ is the Lie superalgebra of all derivations of $\mathbb{C}[t, t^{-1}] \otimes \Lambda(N)$. Let ∂_i stand for $\partial/\partial\theta_i$ and ∂_t stand for $\partial/\partial t$. Every $D \in W(N)$ is represented by a differential operator,

$$D = f\partial_t + \sum_{i=1}^{N} f_i\partial_i, \tag{2.3}$$

where $f, f_i \in \mathbb{C}[t, t^{-1}] \otimes \Lambda(N)$. $W(N)$ has no nontrivial 2-cocycles if $N > 2$. If $N = 1$ or 2, then there exists, up to equivalence, one nontrivial 2-cocycle on $W(N)$.

The superalgebras $S(N, \alpha)$. The Lie superalgebra $W(N)$ contains a one-parameter family of Lie superalgebras $S(N, \alpha)$; see Refs. 15 and 1. By definition

$$S(N, \alpha) = \{D \in W(N) \mid Div(t^\alpha D) = 0\} \text{ for } \alpha \in \mathbb{C}. \tag{2.4}$$

Recall that

$$Div(D) = \partial_t(f) + \sum_{i=1}^{N}(-1)^{p(f_i)}\partial_i(f_i) \tag{2.5}$$

and

$$Div(fD) = Df + fDivD, \tag{2.6}$$

where f is an even function. Let $S'(N,\alpha) = [S(N,\alpha), S(N,\alpha)]$ be the derived superalgebra. Assume that $N > 1$. If $\alpha \notin \mathbb{Z}$, then $S(N,\alpha)$ is simple, and if $\alpha \in \mathbb{Z}$, then $S'(N,\alpha)$ is a simple ideal of $S(N,\alpha)$ of codimension one defined from the exact sequence,

$$0 \to S'(N,\alpha) \to S(N,\alpha) \to \mathbb{C}t^{-\alpha}\theta_1 \cdots \theta_N \partial_t \to 0. \tag{2.7}$$

Notice that

$$S(N,\alpha) \cong S(N,\alpha+n) \text{ for } n \in \mathbb{Z}. \tag{2.8}$$

There exists, up to equivalence, one nontrivial 2-cocycle on $S'(N,\alpha)$ if and only if $N = 2$; see Ref. 1. Let $\hat{S}'(2,\alpha)$ be the corresponding central extension of $S'(2,\alpha)$. Note that $S'(2,\alpha)$ is spanned by 4 even fields and 4 odd fields. Sometimes the name "$N = 4$ superconformal algebra" is used for $\hat{S}'(2,0)$; see Refs. 4 and 3.

The superalgebras $K(N)$. By definition

$$K(N) = \{D \in W(N) \mid D\Omega = f\Omega \text{ for some } f \in \mathbb{C}[t,t^{-1}] \otimes \Lambda(N)\}, \tag{2.9}$$

where

$$\Omega = dt - \sum_{i=1}^{N} \theta_i d\theta_i \tag{2.10}$$

is a contact 1-form; see Refs. 1, 2, 3, and 25. (See also Ref. 26, where the contact superalgebra $K(m,n)$ was introduced, and Ref. 24). Every differential operator $D \in K(N)$ can be represented by a single function,

$$f \in \mathbb{C}[t,t^{-1}] \otimes \Lambda(N) : f \to D_f. \tag{2.11}$$

Let

$$\Delta(f) = 2f - \sum_{i=1}^{N} \theta_i \partial_i(f). \tag{2.12}$$

Then

$$D_f = \Delta(f)\partial_t + \partial_t(f)\sum_{i=1}^{N} \theta_i\partial_i + (-1)^{p(f)}\sum_{i=1}^{N} \partial_i(f)\partial_i. \tag{2.13}$$

Notice that

$$\begin{aligned} D_{f+g} &= D_f + D_g, \\ [D_f, D_g] &= D_{\{f,g\}}, \end{aligned} \tag{2.14}$$

where

$$\{f,g\} = \Delta(f)\partial_t(g) - \partial_t(f)\Delta(g) + (-1)^{p(f)}\sum_{i=1}^{N} \partial_i(f)\partial_i(g). \tag{2.15}$$

The superalgebras $K(N)$ are simple, except when $N = 4$. If $N = 4$, then the derived superalgebra $K'(4) = [K(4), K(4)]$ is a simple ideal in $K(4)$ of codimension one defined from the exact sequence

$$0 \to K'(4) \to K(4) \to \mathbb{C}D_{t^{-1}\theta_1\theta_2\theta_3\theta_4} \to 0. \tag{2.16}$$

There exists no nontrivial 2-cocycles on $K(N)$ if $N > 4$. If $N \leq 3$, then there exists, up to equivalence, one nontrivial 2-cocycle. Let $\hat{K}(N)$ be the corresponding central extension of $K(N)$. Notice that $\hat{K}(1)$ is isomorphic to the Neveu-Schwarz algebra (Ref. 17), and $\hat{K}(2) \cong \hat{W}(1)$ is isomorphic to the so-called $N = 2$ superconformal algebra; see Ref. 18. The superalgebra $K'(4)$ has 3 independent central extensions (Refs. 1, 2, 13 and 14), which is important for our task.

III. Lie superalgebras of pseudodifferential symbols

Recall that the ring R of pseudodifferential symbols is the ring of the formal series

$$A(t, \xi) = \sum_{-\infty}^{n} a_i(t)\xi^i, \tag{3.1}$$

where $a_i(t) \in \mathbb{C}[t, t^{-1}]$, and the variable ξ corresponds to $\partial/\partial t$; see Refs. 9, 10, 11, and 12. The multiplication rule in R is determined as follows:

$$A(t, \xi) \circ B(t, \xi) = \sum_{n \geq 0} \frac{1}{n!} \partial_\xi^n A(t, \xi) \partial_t^n B(t, \xi). \tag{3.2}$$

Notice that R is a generalization of the associative algebra of the regular differential operators on the circle, and the multiplication rule in R, when restricted to the polynomials in ξ, coincides with the multiplication rule for the differential operators. The Lie algebra structure on R is given by

$$[A, B] = A \circ B - B \circ A, \tag{3.3}$$

where $A, B \in R$.

The Poisson algebra P of pseudodifferential symbols has the same underlying vector space. The multiplication in P is naturally defined. The Poisson bracket is defined as follows:

$$\{A(t, \xi), B(t, \xi)\} = \partial_\xi A(t, \xi) \partial_t B(t, \xi) - \partial_t A(t, \xi) \partial_\xi B(t, \xi) \tag{3.4}$$

(Refs. 12 and 19). One can construct the contraction of the Lie algebra R to P using the linear isomorphisms:

$$\varphi_h : R \longrightarrow R \tag{3.5}$$

defined by

$$\varphi_h(a_i(t)\xi^i) = a_i(t)h^i\xi^i, \text{ where } h \in]0,1], \tag{3.6}$$

see Ref. 12. The new multiplication in R is defined by

$$A \circ_h B = \varphi_h^{-1}(\varphi_h(A) \circ \varphi_h(B)). \tag{3.7}$$

Correspondingly, the commutator is

$$[A, B]_h = A \circ_h B - B \circ_h A. \tag{3.8}$$

Thus

$$[A, B]_h = h\{A, B\} + hO(h). \tag{3.9}$$

Hence

$$\lim_{h\to 0} \frac{1}{h}[A, B]_h = \{A, B\}. \tag{3.10}$$

To construct a superanalog of R, consider an associative superalgebra $\Theta_h(N)$ with generators $\theta_1, \ldots, \theta_N, \partial_1, \ldots, \partial_N$ and relations

$$\begin{aligned} \theta_i\theta_j &= -\theta_j\theta_i, \\ \partial_i\partial_j &= -\partial_j\partial_i, \\ \partial_i\theta_j &= h\delta_{i,j} - \theta_j\partial_i, \end{aligned} \tag{3.11}$$

where $h \in]0,1]$. Define an associative superalgebra,

$$R_h(N) = R \otimes \Theta_h(N), \tag{3.12}$$

such that

$$(A \otimes X)(B \otimes Y) = \frac{1}{h}(A \circ_h B) \otimes (XY), \tag{3.13}$$

where $A, B \in R$, and $X, Y \in \Theta_h(N)$. The product in $R_h(N)$ determines the natural Lie superalgebra structure on this space:

$$[(A \otimes X), (B \otimes Y)]_h = \frac{1}{h}(A \circ_h B) \otimes (XY) - (-1)^{p(X)p(Y)}\frac{1}{h}(B \circ_h A) \otimes (YX). \tag{3.14}$$

For each $h \in]0,1]$ there exists an embedding

$$W(N) \subset R_h(N), \tag{3.15}$$

such that the commutation relations in $R_h(N)$, when restricted to $W(N)$, coincide with the commutation relations in $W(N)$. In particular, when $h=1$, we obtain the superanalog $R(N) := R_{h=1}(N)$ of the Lie algebra of pseudodifferential symbols on the circle.

The Poisson superalgebra $P(N)$ has the underlying vector space $P \otimes \Theta(N)$, where $\Theta(N) := \Theta_{h=0}(N)$ is the Grassman algebra with generators $\theta_1, \dots, \theta_N, \bar{\theta}_1, \dots, \bar{\theta}_N$, where $\bar{\theta}_i = \partial_i$ for $i = 1, \dots, N$. The Poisson bracket is defined as follows:

$$\{A, B\} = \partial_\xi A \partial_t B - \partial_t A \partial_\xi B - (-1)^{p(A)} \left(\sum_{i=1}^{N} \partial_{\theta_i} A \partial_{\bar{\theta}_i} B + \partial_{\bar{\theta}_i} A \partial_{\theta_i} B\right), \tag{3.16}$$

where $A, B \in P(N)$; cf. Refs. 2, 5. Thus

$$\lim_{h\to 0} [A, B]_h = \{A, B\}. \tag{3.17}$$

Correspondingly, we have the embedding

$$W(N) \subset P(N). \tag{3.18}$$

Remark 3.1: Recall that there exist, up to equivalence, two nontrivial 2-cocycles on R (Refs. 9, 10, and 11). Analogously, one can define two 2-cocycles, c_ξ and c_t, on $R(N)$; cf. Ref. 20. Let $A, B \in R$, and $X, Y \in \Theta_{h=1}(N)$. Then

$$c_\xi(A \otimes X, B \otimes Y) = \text{ the coefficient of } t^{-1}\xi^{-1}\theta_1 \dots \theta_N \partial_1 \dots \partial_N \tag{3.19}$$

$$\text{in } ([\log \xi, A] \circ B) \otimes (XY),$$

where

$$[\log \xi, A(t, \xi)] = \Sigma_{k \geq 1} \frac{(-1)^{k+1}}{k} \partial_t^k A(t, \xi) \xi^{-k}, \tag{3.20}$$

and

$$c_t(A \otimes X, B \otimes Y) = \text{ the coefficient of } t^{-1}\xi^{-1}\theta_1 \dots \theta_N \partial_1 \dots \partial_N \tag{3.21}$$

$$\text{in } ([\log t, A] \circ B) \otimes (XY),$$

where

$$[\log t, A(t, \xi)] = \Sigma_{k \geq 1} \frac{(-1)^{k+1}}{k} t^{-k} \partial_\xi^k A(t, \xi). \tag{3.22}$$

IV. The construction of embedding

Let $DerS'(2,\alpha)$ be the Lie superalgebra of all derivations of $S'(2,\alpha)$.

Lemma 4.1: The exterior derivations $Der_{ext}S'(2,\alpha)$ for all $\alpha\in\mathbb{Z}$ generate the loop algebra

$$\tilde{\mathfrak{sl}}(2)\subset P(2). \tag{4.1}$$

Proof: In Ref. 21 we observed that the exterior derivations of $S'(2,0)$ form an $\mathfrak{sl}(2)$. Let

$$\{\mathfrak{L}_n^\alpha, E_n, H_n, F_n, \mathfrak{h}_n^\alpha, \mathfrak{p}_n^0, \mathfrak{x}_n^0, \mathfrak{y}_n^\alpha\}_{n\in\mathbb{Z}} \tag{4.2}$$

be a basis of $S'(2,\alpha)$ defined as follows:

$$\begin{aligned}
\mathfrak{L}_n^\alpha &= -t^n(t\xi+\frac{1}{2}(n+\alpha+1)(\theta_1\partial_1+\theta_2\partial_2)), \qquad (4.3)\\
E_n &= t^n\theta_2\partial_1,\\
H_n &= t^n(\theta_2\partial_2-\theta_1\partial_1),\\
F_n &= t^n\theta_1\partial_2,\\
\mathfrak{h}_n^\alpha &= t^n\xi\theta_2-(n+\alpha)t^{n-1}\theta_1\theta_2\partial_1,\\
\mathfrak{p}_n^0 &= -t^{n+1}\partial_2,\\
\mathfrak{x}_n^0 &= t^{n+1}\partial_1,\\
\mathfrak{y}_n^\alpha &= t^n\xi\theta_1+(n+\alpha)t^{n-1}\theta_1\theta_2\partial_2.
\end{aligned}$$

Let us show that if $\alpha\in\mathbb{Z}$, then $Der_{ext}S'(2,\alpha)\cong\mathfrak{sl}(2)=\langle\boldsymbol{\mathcal{E}},\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{F}}\rangle$, where

$$[\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{E}}]=2\boldsymbol{\mathcal{E}}, [\boldsymbol{\mathcal{H}},\boldsymbol{\mathcal{F}}]=-2\boldsymbol{\mathcal{F}}, [\boldsymbol{\mathcal{E}},\boldsymbol{\mathcal{F}}]=\boldsymbol{\mathcal{H}}, \tag{4.4}$$

and the action of $\mathfrak{sl}(2)$ is given as follows:

$$\begin{aligned}
&[\boldsymbol{\mathcal{E}},\mathfrak{h}_n^\alpha]=\mathfrak{x}_{n-1+\alpha}, [\boldsymbol{\mathcal{E}},\mathfrak{y}_n^\alpha]=\mathfrak{p}_{n-1+\alpha}^0, [\boldsymbol{\mathcal{F}},\mathfrak{x}_n]=\mathfrak{h}_{n+1-\alpha}^\alpha, [\boldsymbol{\mathcal{F}},\mathfrak{p}_n^0]=\mathfrak{y}_{n+1-\alpha}^\alpha, \qquad (4.5)\\
&[\boldsymbol{\mathcal{H}},\mathfrak{x}_n^0]=\mathfrak{x}_n^0, [\boldsymbol{\mathcal{H}},\mathfrak{h}_n^\alpha]=-\mathfrak{h}_n^\alpha, [\boldsymbol{\mathcal{H}},\mathfrak{p}_n^0]=\mathfrak{p}_n^0, [\boldsymbol{\mathcal{H}},\mathfrak{y}_n^\alpha]=-\mathfrak{y}_n^\alpha.
\end{aligned}$$

Notice that

$$Der_{ext}S'(2,\alpha)\cong H^1(S'(2,\alpha),S'(2,\alpha)), \tag{4.6}$$

see Ref. 22. Consider the following $\mathbb{Z}$-grading deg of $S'(2,\alpha)$:

$$\begin{aligned}
&\deg\,\mathfrak{L}_n^\alpha=n, \deg\,E_n=n+1-\alpha, \deg\,F_n=n-1+\alpha, \deg\,H_n=n, \qquad (4.7)\\
&\deg\,\mathfrak{h}_n^\alpha=n, \deg\,\mathfrak{p}_n=n, \deg\,\mathfrak{x}_n=n+1-\alpha, \deg\,\mathfrak{y}_n^\alpha=n-1+\alpha.
\end{aligned}$$

Let

$$L_0^\alpha = -\mathcal{L}_0^\alpha + \frac{1}{2}(1-\alpha)H_0. \tag{4.8}$$

Then

$$[L_0^\alpha, s] = (\deg\, s)s \tag{4.9}$$

for a homogeneous $s \in S'(2,\alpha)$. Accordingly,

$$[L_0^\alpha, D] = (\deg\, D)D \tag{4.10}$$

for a homogeneous $D \in Der_{ext}S'(2,\alpha)$. On the other hand, since the action of a Lie superalgebra on its cohomology is trivial, then one must have

$$[L_0^\alpha, D] = 0. \tag{4.11}$$

Hence the nonzero elements of $Der_{ext}S'(2,\alpha)$ have deg $= 0$, and they preserve the superalgebra $S'(2,\alpha)_{\text{deg}=0}$. One can check that the exterior derivations of $S'(2,\alpha)_{\text{deg}=0}$ form an $\mathfrak{sl}(2)$, and extend them to the exterior derivations of $S'(2,\alpha)$ as in (4.5). One should also note that if the restriction of a derivation of $S'(2,\alpha)$ to $S'(2,\alpha)_{\text{deg}=0}$ is zero, then this derivation is inner. We can identify the exterior derivation $t^{-\alpha}\xi\theta_1\theta_2$ [see (2.7)] with $-\boldsymbol{\mathcal{F}}$. We cannot realize all the exterior derivations as regular differential operators on the supercircle, but can do this using the symbols of pseudodifferential operators. In fact, let $\alpha = 1$. Then

$$Der_{ext}S'(2,1) = \mathfrak{sl}(2) = \langle \boldsymbol{\mathcal{F}}, \boldsymbol{\mathcal{H}}, \boldsymbol{\mathcal{E}} \rangle \subset P(2), \tag{4.12}$$

where

$$\boldsymbol{\mathcal{F}} = -t^{-1}\xi\theta_1\theta_2, \boldsymbol{\mathcal{H}} = -\theta_1\partial_1 - \theta_2\partial_2, \boldsymbol{\mathcal{E}} = t\xi^{-1}\partial_1\partial_2. \tag{4.13}$$

One can then construct the loop algebra of $\mathfrak{sl}(2)$ as follows:

$$\tilde{\mathfrak{sl}}(2) = \langle \boldsymbol{\mathcal{F}}_n, \boldsymbol{\mathcal{H}}_n, \boldsymbol{\mathcal{E}}_n \rangle_{n\in\mathbb{Z}}, \tag{4.14}$$

where

$$\begin{aligned} \boldsymbol{\mathcal{F}}_n &= -t^{n-1}\xi\theta_1\theta_2, \\ \boldsymbol{\mathcal{H}}_n &= nt^{n-1}\xi^{-1}\theta_1\theta_2\partial_1\partial_2 - t^n(\theta_1\partial_1 + \theta_2\partial_2), \\ \boldsymbol{\mathcal{E}}_n &= t^{n+1}\xi^{-1}\partial_1\partial_2. \end{aligned} \tag{4.15}$$

The nonvanishing commutation relations are

$$[\boldsymbol{\mathcal{H}}_n, \boldsymbol{\mathcal{E}}_k] = 2\boldsymbol{\mathcal{E}}_{n+k}, [\boldsymbol{\mathcal{H}}_n, \boldsymbol{\mathcal{F}}_k] = -2\boldsymbol{\mathcal{F}}_{n+k}, [\boldsymbol{\mathcal{E}}_n, \boldsymbol{\mathcal{F}}_k] = \boldsymbol{\mathcal{H}}_{n+k}. \tag{4.16}$$

Let $\alpha \in \mathbb{Z}$. Then

$$Der_{ext}S'(2,\alpha) \cong \langle \boldsymbol{\mathcal{F}}_{-\alpha+1}, \boldsymbol{\mathcal{H}}_0, \boldsymbol{\mathcal{E}}_{\alpha-1} \rangle. \tag{4.17}$$

$\square$

Theorem 4.1: The superalgebras $S'(2,\alpha)$ for all $\alpha \in \mathbb{Z}$ together with $\tilde{\mathfrak{sl}}(2)$ generate a Lie superalgebra isomorphic to $K'(4)$.

Proof: Let

$$\begin{aligned} I_n^0 &= t^n(\theta_1\partial_1 + \theta_2\partial_2), \\ \mathfrak{r}_n &= t^{n-1}\theta_1\theta_2\partial_1, \\ \mathfrak{s}_n &= t^{n-1}\theta_1\theta_2\partial_2. \end{aligned} \tag{4.18}$$

Then according to (4.3)

$$\begin{aligned} \mathfrak{L}_n^\alpha &= \mathfrak{L}_n^0 - \frac{1}{2}\alpha I_n^0, \\ \mathfrak{h}_n^\alpha &= \mathfrak{h}_n^0 - \alpha\mathfrak{r}_n, \\ \mathfrak{y}_n^\alpha &= \mathfrak{y}_n^0 + \alpha\mathfrak{s}_n. \end{aligned} \tag{4.19}$$

One can easily check that the superalgebras $S'(2,\alpha)$, where $\alpha \in \mathbb{Z}$, generate $W(2) \subset P(2)$. In fact, $W(2)$ is spanned by 8 fields defined in Eq. (4.3), where $\alpha = 0$, together with 3 fields defined in Eq. (4.18) and the field $\boldsymbol{\mathcal{F}}_n$. If we include two even fields, $\boldsymbol{\mathcal{E}}_n$ and $\boldsymbol{\mathcal{H}}_n$, into the picture, then from the commutation relations, we obtain two additional odd fields:

$$\begin{aligned} \mathfrak{q}_n &= t^n\xi^{-1}\theta_2\partial_1\partial_2, \\ \mathfrak{t}_n &= -t^n\xi^{-1}\theta_1\partial_1\partial_2. \end{aligned} \tag{4.20}$$

Let $\mathfrak{g} \subset P(2)$ be the Lie superalgebra generated by the superalgebras $S'(2,\alpha)$ for all $\alpha \in \mathbb{Z}$ and $\tilde{\mathfrak{sl}}(2)$. We will show that there exists an isomorphism:

$$\psi : K'(4) \longrightarrow \mathfrak{g}. \tag{4.21}$$

Let

$$\begin{aligned} \mathfrak{L}_n &= \mathfrak{L}_n^0 + \boldsymbol{\mathcal{H}}_n + \frac{1}{2}I_n^0, \\ I_n &= I_n^0 + \boldsymbol{\mathcal{H}}_n, \\ \mathfrak{p}_n &= \mathfrak{p}_n^0 + \mathfrak{t}_n, \\ \mathfrak{x}_n &= \mathfrak{x}_n^0 - \mathfrak{q}_n. \end{aligned} \tag{4.22}$$

Set

$$\mathfrak{h}_n = \mathfrak{h}_n^0, \mathfrak{y}_n = \mathfrak{y}_n^0. \tag{4.23}$$

Then $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$, where

$$\begin{aligned} \mathfrak{g}_{\bar{0}} &= \langle \mathfrak{L}_n, I_n, E_n, H_n, F_n, \boldsymbol{\mathcal{E}}_n, \boldsymbol{\mathcal{H}}_n, \boldsymbol{\mathcal{F}}_n \rangle, \\ \mathfrak{g}_{\bar{1}} &= \langle \mathfrak{h}_n, \mathfrak{p}_n, \mathfrak{x}_n, \mathfrak{y}_n, \mathfrak{r}_n, \mathfrak{s}_n, \mathfrak{q}_n, \mathfrak{t}_n \rangle. \end{aligned} \tag{4.24}$$

We will describe the nonvanishing commutation relations in $\mathfrak{g}$ with respect to this basis.

For $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{0}}]$ the relations are:

$[\mathfrak{L}_n, \mathfrak{L}_k] = (n-k)\mathfrak{L}_{n+k}$; (4.25)
$[H_n, E_k] = 2E_{n+k}, [H_n, F_k] = -2F_{n+k}, [E_n, F_k] = H_{n+k}$;
$[\boldsymbol{\mathcal{H}}_n, \boldsymbol{\mathcal{E}}_k] = 2\boldsymbol{\mathcal{E}}_{n+k}, [\boldsymbol{\mathcal{H}}_n, \boldsymbol{\mathcal{F}}_k] = -2\boldsymbol{\mathcal{F}}_{n+k}, [\boldsymbol{\mathcal{E}}_n, \boldsymbol{\mathcal{F}}_k] = \boldsymbol{\mathcal{H}}_{n+k}$;
$[\mathfrak{L}_n, X_k] = -kX_{n+k}$, where $X_k = I_k, E_k, H_k, F_k, \boldsymbol{\mathcal{E}}_k, \boldsymbol{\mathcal{H}}_k, \boldsymbol{\mathcal{F}}_k$.

For $[\mathfrak{g}_{\bar{0}}, \mathfrak{g}_{\bar{1}}]$ the relations are:

$[\mathfrak{L}_n, X_k] = (-k + \frac{n}{2})X_{n+k}$, where $X_k = \mathfrak{h}_k, \mathfrak{p}_k, \mathfrak{x}_k, \mathfrak{y}_k$; (4.26)
$[\mathfrak{L}_n, X_k] = (-k - \frac{n}{2})X_{n+k}$, where $X_k = \mathfrak{r}_k, \mathfrak{s}_k, \mathfrak{q}_k, \mathfrak{t}_k$;
$[I_n, X_k] = nY_{n+k}$, where $X_k = \mathfrak{h}_k, \mathfrak{p}_k, \mathfrak{x}_k, \mathfrak{y}_k$, and $Y_k = \mathfrak{r}_k, \mathfrak{t}_k, -\mathfrak{q}_k, -\mathfrak{s}_k$, respectively;
$[H_n, X_k] = X_{n+k}$, where $X_k = \mathfrak{h}_k, \mathfrak{x}_k, \mathfrak{q}_k, \mathfrak{r}_k$;
$[H_n, X_k] = -X_{n+k}$, where $X_k = \mathfrak{y}_k, \mathfrak{p}_k, \mathfrak{s}_k, \mathfrak{t}_k$;
$[E_n, X_k] = Y_{n+k}, [F_n, Y_k] = X_{n+k}$,
where $X_k = \mathfrak{y}_k, \mathfrak{p}_k, \mathfrak{s}_k, \mathfrak{t}_k$, and $Y_k = \mathfrak{h}_k, \mathfrak{x}_k, -\mathfrak{r}_k, -\mathfrak{q}_k$, respectively;
$[\boldsymbol{\mathcal{H}}_n, X_k] = X_{n+k} + nY_{n+k}$,
where $X_k = \mathfrak{p}_k, \mathfrak{x}_k, \mathfrak{q}_k, \mathfrak{t}_k$, and $Y_k = \mathfrak{t}_k, -\mathfrak{q}_k, 0, 0$, respectively;
$[\boldsymbol{\mathcal{H}}_n, X_k] = -X_{n+k} - nY_{n+k}$,
where $X_k = \mathfrak{h}_k, \mathfrak{y}_k, \mathfrak{r}_k, \mathfrak{s}_k$, and $Y_k = \mathfrak{r}_k, -\mathfrak{s}_k, 0, 0$, respectively;
$[\boldsymbol{\mathcal{E}}_n, X_k] = Y_{n+k} - nZ_{n+k}$, $[\boldsymbol{\mathcal{F}}_n, Y_k] = X_{n+k} - n\bar{Z}_{n+k}$, where $X_k = \mathfrak{h}_k, \mathfrak{y}_k, \mathfrak{r}_k, \mathfrak{s}_k$,
$Y_k = \mathfrak{x}_k, \mathfrak{p}_k, -\mathfrak{q}_k, -\mathfrak{t}_k$, $Z_k = \mathfrak{q}_k, -\mathfrak{t}_k, 0, 0$, and $\bar{Z}_k = -\mathfrak{r}_k, \mathfrak{s}_k, 0, 0$, respectively.

Finally, for $[\mathfrak{g}_{\bar{1}}, \mathfrak{g}_{\bar{1}}]$ the relations are:

$[\mathfrak{h}_n, \mathfrak{x}_k] = (k-n)E_{n+k}, [\mathfrak{p}_n, \mathfrak{y}_k] = (k-n)F_{n+k}$, (4.27)
$[\mathfrak{h}_n, \mathfrak{p}_k] = \mathfrak{L}_{n+k} - \frac{1}{2}(k-n)H_{n+k}, [\mathfrak{x}_n, \mathfrak{y}_k] = -\mathfrak{L}_{n+k} + \frac{1}{2}(k-n)H_{n+k}$,
$[\mathfrak{h}_n, \mathfrak{q}_k] = E_{n+k}, [\mathfrak{x}_n, \mathfrak{r}_k] = E_{n+k}$, $[\mathfrak{p}_n, \mathfrak{s}_k] = F_{n+k}, [\mathfrak{y}_n, \mathfrak{t}_k] = F_{n+k}$,
$[\mathfrak{p}_n, \mathfrak{q}_k] = -\boldsymbol{\mathcal{E}}_{n+k}, [\mathfrak{x}_n, \mathfrak{t}_k] = -\boldsymbol{\mathcal{E}}_{n+k}, [\mathfrak{h}_n, \mathfrak{s}_k] = -\boldsymbol{\mathcal{F}}_{n+k}, [\mathfrak{y}_n, \mathfrak{r}_k] = -\boldsymbol{\mathcal{F}}_{n+k}$,
$[\mathfrak{p}_n, \mathfrak{r}_k] = \frac{1}{2}I_{n+k} - \frac{1}{2}(H_{n+k} + \boldsymbol{\mathcal{H}}_{n+k}), [\mathfrak{x}_n, \mathfrak{s}_k] = \frac{1}{2}I_{n+k} + \frac{1}{2}(H_{n+k} - \boldsymbol{\mathcal{H}}_{n+k})$,
$[\mathfrak{h}_n, \mathfrak{t}_k] = \frac{1}{2}I_{n+k} + \frac{1}{2}(H_{n+k} + \boldsymbol{\mathcal{H}}_{n+k}), [\mathfrak{y}_n, \mathfrak{q}_k] = \frac{1}{2}I_{n+k} - \frac{1}{2}(H_{n+k} - \boldsymbol{\mathcal{H}}_{n+k})$.

Recall that the elements of $K(4)$ can be identified with the functions from $\mathbb{C}[t,t^{-1}] \otimes \Lambda(4)$. Let

$$\check{\theta}_1 = \theta_2\theta_3\theta_4, \check{\theta}_2 = \theta_1\theta_3\theta_4, \check{\theta}_3 = \theta_1\theta_2\theta_4, \check{\theta}_4 = \theta_1\theta_2\theta_3. \tag{4.28}$$

The following 16 series of functions together with $t^{-1}\theta_1\theta_2\theta_3\theta_4$ span $\mathbb{C}[t,t^{-1}] \otimes \Lambda(4)$:

$$\begin{aligned}
f_n^1 &= 2nt^{n-1}\theta_1\theta_2\theta_3\theta_4, \\
f_n^2 &= -\frac{1}{2}t^{n+1} + \frac{1}{2}it^n(\theta_2\theta_3 - \theta_1\theta_4) - \frac{1}{2}n(n+1)t^{n-1}\theta_1\theta_2\theta_3\theta_4, \\
f_n^k &= \frac{1}{2}t^{n\mp 1}(\pm\theta_1\theta_2 \mp \theta_3\theta_4 - i\theta_1\theta_3 - i\theta_2\theta_4), k = 3, 4, \\
f_n^5 &= it^n(\theta_1\theta_4 - \theta_2\theta_3), \\
f_n^k &= \frac{1}{2}t^n(\mp\theta_1\theta_4 \mp \theta_2\theta_3 + i\theta_2\theta_4 - i\theta_1\theta_3), k = 6, 7, \\
f_n^8 &= -it^n(\theta_1\theta_2 + \theta_3\theta_4), \\
f_n^k &= \frac{(i)^{p(k)}}{\sqrt{8}}\big(t^n(\theta_1 \mp i\theta_2 \mp \theta_3 + i\theta_4) - nt^{n-1}(\check{\theta}_1 \pm i\check{\theta}_2 \mp \check{\theta}_3 - i\check{\theta}_4)\big), k = 9, 10, \\
f_n^k &= \frac{(i)^{p(k)}}{\sqrt{8}}\big(t^{n+1}(\theta_1 \pm i\theta_2 \mp \theta_3 - i\theta_4) - (n+1)t^n(\check{\theta}_1 \mp i\check{\theta}_2 \mp \check{\theta}_3 + i\check{\theta}_4)\big), k = 11, 12, \\
f_n^k &= \frac{(-i)^{p(k)}}{\sqrt{2}}t^{n-1}(\check{\theta}_1 \pm i\check{\theta}_2 \mp \check{\theta}_3 - i\check{\theta}_4), k = 13, 14, \\
f_n^k &= \frac{(-i)^{p(k)}}{\sqrt{2}}t^n(\check{\theta}_1 \mp i\check{\theta}_2 \mp \check{\theta}_3 + i\check{\theta}_4), k = 15, 16,
\end{aligned} \tag{4.29}$$

where $p(k) = 0$ if k is even, and $p(k) = 1$ if k is odd.

The 16 series of the corresponding differential operators $\{D_{f_n^i}\}_{i=1,\ldots,16}$ span $K'(4)$. Set

$$\begin{aligned}
&\psi(D_{f_n^1}) = I_n, \psi(D_{f_n^2}) = \mathfrak{L}_n, \\
&\psi(D_{f_n^3}) = E_n, \psi(D_{f_n^4}) = F_n, \psi(D_{f_n^5}) = H_n, \\
&\psi(D_{f_n^6}) = \boldsymbol{\mathcal{E}}_n, \psi(D_{f_n^7}) = \boldsymbol{\mathcal{F}}_n, \psi(D_{f_n^8}) = \boldsymbol{\mathcal{H}}_n, \\
&\psi(D_{f_n^9}) = \mathfrak{x}_n, \psi(D_{f_n^{10}}) = \mathfrak{h}_n, \psi(D_{f_n^{11}}) = \mathfrak{y}_n, \psi(D_{f_n^{12}}) = \mathfrak{p}_n, \\
&\psi(D_{f_n^{13}}) = \mathfrak{q}_n, \psi(D_{f_n^{14}}) = \mathfrak{r}_n, \psi(D_{f_n^{15}}) = \mathfrak{s}_n, \psi(D_{f_n^{16}}) = \mathfrak{t}_n.
\end{aligned} \tag{4.30}$$

Notice that $f_n^1 = 0$, if $n = 0$. This corresponds to the fact that $D_{t^{-1}\theta_1\theta_2\theta_3\theta_4} \notin K'(4)$. One can verify that ψ is an isomorphism from $K'(4)$ onto $\mathfrak{g}$.

$\square$

Remark 4.2: We have obtained an embedding

$$K'(4) \subset P(2). \tag{4.31}$$

In general, a Lie algebra of contact vector fields can be realized as a subalgebra of Poisson algebra; see Ref. 8. We will explain this from the geometrical point of view in application to our case. Recall that the Lie algebra $Vect(S^1)$ of smooth vector fields on the circle has a natural embedding into the Poisson algebra of functions on the cylinder $\dot{T}^*S^1 = T^*S^1 \setminus S^1$ with the removed zero section; see Refs. 11, 12 and 19. One can introduce the Darboux coordinates $(q,p) = (t,\xi)$ on this manifold. The symbols of differential operators are functions on $\dot{T}^*S^1$ which are formal Laurent series in p with coefficients periodic in q. Correspondingly, they define Hamiltonian vector fields on $\dot{T}^*S^1$:

$$A(q,p) \longrightarrow H_A = \partial_p A \partial_q - \partial_q A \partial_p. \tag{4.32}$$

The embedding of $Vect(S^1)$ into the Lie algebra of Hamiltonian vector fields on $\dot{T}^*S^1$ is given by

$$f(q)\partial_q \longrightarrow H_{f(q)p}. \tag{4.33}$$

Notice that we obtain a subalgebra of Hamiltonian vector fields with Hamiltonians which are homogeneous of degree 1. (This condition holds in general, if one considers the *symplectification* of a contact manifold; see Ref. 8.) In other words, we obtain a subalgebra of Hamiltonian vector fields, which commute with the (semi-) Euler vector field:

$$[H_A, p\partial_p] = 0. \tag{4.34}$$

We will show that for $N \geq 0$ there exists the analogous embedding:

$$K(2N) \subset P(N). \tag{4.35}$$

The analog of the formula (4.32) in the supercase is as follows (Refs. 2, 5):

$$A(q,p,\theta_i,\bar{\theta}_i) \longrightarrow H_A = \partial_p A \partial_q - \partial_q A \partial_p - (-1)^{p(A)} \sum_{i=1}^{N} (\partial_{\theta_i} A \partial_{\bar{\theta}_i} + \partial_{\bar{\theta}_i} A \partial_{\theta_i}). \tag{4.36}$$

Then $K(2N)$ is defined as the set of all (Hamiltonian) functions $A(q,p,\theta_i,\bar{\theta}_i) \in P(N)$ such that

$$[H_A, p\partial_p + \sum_{i=1}^{N} \bar{\theta}_i \partial_{\bar{\theta}_i}] = 0. \tag{4.37}$$

Equivalently, we have the following characterization of the embedding (4.35). Consider a $\mathbb{Z}$-grading of the (associative) superalgebra $P(N) = \oplus_{j\in\mathbb{Z}} P_j(N)$ defined by

$$\begin{aligned} &\deg p = \deg \bar{\theta}_i = 1 \text{ for } i = 1, \ldots, N, \\ &\deg q = \deg \theta_i = 0 \text{ for } i = 1, \ldots, N. \end{aligned} \tag{4.38}$$

Thus with respect to the Poisson bracket,

$$\{P_j(N), P_k(N)\} \subset P_{j+k-1}(N). \tag{4.39}$$

Then

$$K(2N) = P_1(N). \tag{4.40}$$

Theorem 4.2: There exists an embedding,

$$\hat{K}'(4) \subset R_h(2), \tag{4.41}$$

for each $h \in]0,1]$, such that the central element in $\hat{K}'(4)$ is $h \in R_h(2)$, and

$$\lim_{h\to 0} \hat{K}'(4) = K'(4) \subset P(2). \tag{4.42}$$

Proof: For each $h \in]0,1]$ and $\alpha \in \mathbb{Z}$ we have an embedding,

$$DerS'(2,\alpha) \subset R_h(2). \tag{4.43}$$

The exterior derivations $Der_{ext}S'(2,\alpha)$ for all $\alpha \in \mathbb{Z}$ generate the loop algebra,

$$\tilde{\mathfrak{sl}}(2) = \langle \boldsymbol{\mathcal{F}}_n, \boldsymbol{\mathcal{H}}_n, \boldsymbol{\mathcal{E}}_n \rangle_{n\in\mathbb{Z}} \subset R_h(2), \text{ where} \tag{4.44}$$

$$\begin{aligned} \boldsymbol{\mathcal{F}}_n &= -t^{n-1}\xi\theta_1\theta_2, \\ \boldsymbol{\mathcal{H}}_n &= \frac{1}{h}((\xi^{-1} \circ_h t^n\xi)(h^2 - h\theta_1\partial_1 - h\theta_2\partial_2 - \theta_1\theta_2\partial_1\partial_2) + t^n\theta_1\theta_2\partial_1\partial_2), \\ \boldsymbol{\mathcal{E}}_n &= (\xi^{-1} \circ_h t^{n+1})\partial_1\partial_2, \end{aligned} \tag{4.45}$$

so that Eqs. (4.16)-(4.17) hold. Let $\mathfrak{g} \subset R_h(2)$ be the Lie superalgebra generated by $S'(2,\alpha)$ for all $\alpha \in \mathbb{Z}$ and $\tilde{\mathfrak{sl}}(2)$. Set

$$\begin{aligned} \mathfrak{q}_n &= (\xi^{-1} \circ_h t^n)(h\partial_1 + \theta_2\partial_1\partial_2), \\ \mathfrak{t}_n &= (\xi^{-1} \circ_h t^n)(h\partial_2 - \theta_1\partial_1\partial_2). \end{aligned} \tag{4.46}$$

The basis (4.24) in $\mathfrak{g}$ is defined by Eqs. (4.3), (4.18), (4.22)-(4.23) and (4.45)-(4.46). The commutation relations in $\mathfrak{g}$ with respect to this basis are given by Eqs. (4.25)-(4.27). The Lie superalgebra $\mathfrak{g}$ is isomorphic to a central extension,

$$\hat{K}'(4) = K'(4) \oplus \mathbb{C}C \tag{4.47}$$

of $K'(4)$. The corresponding 2-cocycle (up to equivalence) is

$$\begin{aligned} c(t^{n+1}, t^{k+1}\theta_1\theta_2\theta_3\theta_4) &= \delta_{n+k+2,0}, \\ c(t^{n+1}\theta_i, t^{k+1}\partial_i(\theta_1\theta_2\theta_3\theta_4)) &= \frac{1}{2}\delta_{n+k+2,0} \text{ for } i = 1, \ldots, 4. \end{aligned} \tag{4.48}$$

The isomorphism,

$$\psi : \hat{K}'(4) \longrightarrow \mathfrak{g} \tag{4.49}$$

is defined by Eq. (4.30) and the equation

$$\psi(C) = I_0 = h \in R_h(2). \tag{4.50}$$

The corresponding 2-cocycle in the basis (4.24) is

$$\begin{aligned} c(\mathsf{p}_n, \mathfrak{r}_k) &= \frac{1}{2}\delta_{n,-k}, \\ c(\mathsf{x}_n, \mathfrak{s}_k) &= \frac{1}{2}\delta_{n,-k}, \\ c(\mathsf{h}_n, \mathsf{t}_k) &= \frac{1}{2}\delta_{n,-k}, \\ c(\mathsf{y}_n, \mathsf{q}_k) &= \frac{1}{2}\delta_{n,-k}, \\ c(\mathcal{L}_n, I_k) &= n\delta_{n,-k}. \end{aligned} \tag{4.51}$$

Note that in the realization of $K'(4)$ inside $P(2)$, obtained in Theorem 4.1, we have $I_0 = 0$.

□

Remark 4.3: The 2-cocycle c is one of three nontrivial 2-cocycles on $K'(4)$; see Refs. 1 and 2. [In Ref. 1 this cocycle is defined by Eq. (4.22), where $d = 0, e = 1$]. Note that the cocycle c is equivalent to the restriction of the 2-cocycle c_t on $R(2)$; see Eqs. (3.21), (3.22).

V. One-parameter family of representations of $\hat{K}'(4)$

Theorem 5.1: There exists a one-parameter family of irreducible representations of $\hat{K}'(4)$ depending on parameter $\mu \in \mathbb{C}$ in the superspace spanned by 2 even fields and 2 odd fields where the value of the central charge is equal to one.

Proof: Let $g \in t^{\mu}\mathbb{C}[t, t^{-1}]$, where $\mu \in \mathbb{R} \setminus \mathbb{Z}$. One can think of ξ^{-1} as the anti-derivative,

$$\xi^{-1} g(t) = \int g(t) dt. \tag{5.1}$$

Let $f(t) \in \mathbb{C}[t, t^{-1}]$. According to (3.2),

$$\xi^{-1} \circ f = \sum_{n=0}^{\infty} (-1)^n (\xi^n f) \xi^{-n-1}. \tag{5.2}$$

Notice that this formula, when applied to a function g, corresponds to the formula of integration by parts. Let

$$V^{\mu} = t^{\mu}\mathbb{C}[t, t^{-1}] \otimes \Lambda(2) = t^{\mu}\mathbb{C}[t, t^{-1}] \otimes \langle 1, \theta_1, \theta_2, \theta_1\theta_2 \rangle, \ \mu \in \mathbb{R} \setminus \mathbb{Z}. \tag{5.3}$$

Using the realization of $\hat{K}'(4)$ inside $R(2)$ (see Theorem 4.2 for $h = 1$) we obtain a representation of $\hat{K}'(4)$ in V^{μ}. A central element in $\hat{K}'(4)$ is $I_0 = 1 \in R(2)$; the 2-cocycle is defined by Eq. (4.51). Let $\{v^i_m\}$, where $m \in \mathbb{Z}$ and $i = 0, 1, 2, 3$, be the following basis in V^{μ}:

$$\begin{aligned} v^0_m &= \frac{1}{m+\mu} t^{m+\mu}, \\ v^1_m &= t^{m+\mu}\theta_1, \\ v^2_m &= t^{m+\mu}\theta_2, \\ v^3_m &= t^{m+\mu}\theta_1\theta_2. \end{aligned} \tag{5.4}$$

The action of $\hat{K}'(4)$ is given as follows:

$$\begin{aligned} \mathfrak{L}_n(v^0_m) &= -(m+n+\mu-1)v^0_{m+n}, \\ \mathfrak{L}_n(v^i_m) &= -(m+\frac{1}{2}n+\mu)v^i_{m+n}, i = 1, 2, \\ \mathfrak{L}_n(v^3_m) &= -(m+n+\mu+1)v^3_{m+n}, \\ E_n(v^1_m) &= v^2_{m+n}, F_n(v^2_m) = v^1_{m+n}, \\ \boldsymbol{\mathcal{E}}_n(v^3_m) &= v^0_{m+n+2}, \boldsymbol{\mathcal{F}}_n(v^0_m) = -v^3_{m+n-2}, \\ H_n(v^i_m) &= \mp v^i_{m+n}, i = 1, 2, \\ \boldsymbol{\mathcal{H}}_n(v^i_m) &= \pm v^i_{m+n}, i = 0, 3, \end{aligned} \tag{5.5}$$

$$
\begin{aligned}
&\mathfrak{h}_n(v^1_m) = -(m+n+\mu)v^3_{m+n-1}, \mathfrak{y}_n(v^2_m) = (m+n+\mu)v^3_{m+n-1},\\
&\mathfrak{h}_n(v^0_m) = v^2_{m+n-1}, \mathfrak{y}_n(v^0_m) = v^1_{m+n-1},\\
&\mathfrak{x}_n(v^1_m) = (m+n+\mu)v^0_{m+n+1}, \mathfrak{p}_n(v^2_m) = -(m+n+\mu)v^0_{m+n+1},\\
&\mathfrak{x}_n(v^3_m) = v^2_{m+n+1}, \mathfrak{p}_n(v^3_m) = v^1_{m+n+1},\\
&\mathfrak{r}_n(v^1_m) = v^3_{m+n-1}, \mathfrak{s}_n(v^2_m) = v^3_{m+n-1},\\
&\mathfrak{q}_n(v^1_m) = v^0_{m+n+1}, \mathfrak{t}_n(v^2_m) = v^0_{m+n+1},\\
&I_n(v^i_m) = v^i_{m+n}, i = 0,1,2,3.
\end{aligned}
$$

Note that I_0 acts by the identity operator. One can then define a one-parameter family of representations of $\hat{K}'(4)$ depending on parameter $\mu \in \mathbb{C}$ in the superspace $V = \langle v^0_m, v^3_m, v^1_m, v^2_m\rangle_{m\in\mathbb{Z}}$, where $p(v^i_m) = \bar{0}$, for $i = 0, 3$, and $p(v^i_m) = \bar{1}$ for $i = 1, 2$, according to the formulas (5.5).

□

Remark 5.1: The elements $\{\mathfrak{L}_n, H_n, \mathfrak{h}_n, \mathfrak{p}_n\}_{n\in\mathbb{Z}}$ span a subalgebra of $K'(4)$ isomorphic to $K(2)$. Note that V decomposes into the direct sum of two submodules over this superalgebra:

$$V = \langle v^0_m, v^2_m\rangle_{m\in\mathbb{Z}} \oplus \langle v^3_m, v^1_m\rangle_{m\in\mathbb{Z}}. \tag{5.6}$$

Remark 5.2: We conjecture that there exists a *two*-parameter family of representations of $\hat{K}'(4)$ in the superspace spanned by 4 fields. In order to define it, instead of the superspace of functions, V^μ, one should consider the superspace of "densities".

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